LOCAL LIPSCHITZ GEOMETRY OF WEIGHTED HOMOGENEOUS SURFACES

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Abstract. We compute Hoelder Complexes,i.e. the complete bi-Lipschitz invariants, for germs of real weighed homogeneous algebraic or semialgebraic surfaces.

1. Introduction

A basic question of Metric Theory of Singularities is Lipschitz Classification of Singular Sets. Some recent results of several authors are devoted to Lipschitz invariants of semialgebraic or algebraic sets with singularities. (See, for example, [1],[3],[4],[5],[7]).

Hölder Complexes, constructed in [1], are complete bi-Lipschitz invariants for germs of semialgebraic surfaces. Lê Dung Trang asked the following natural question: what is the relation between Lipschitz invariants and the algebraic nature of the semialgebraic sets? In this paper we give a complete answer to this question for weighted homogeneous surfaces in $\mathbb{R}^n$, i.e. we compute the exponents in Hölder Complexes of these sets.

In order to compute these exponents we consider weighted homogeneous singular foliations in $\mathbb{R}^n$ and prove that the corresponding Hölder Exponents can be computed in terms of orders of contact of leaves of such a foliation.

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2. Preliminaries and main results

We are going to recall a definition of Canonical Hölder Complex presented in [1].

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An Abstract Hölder Complex is a pair (Γ, β), where Γ is a finite graph, E_Γ is the set of edges of Γ and $\beta\colon E_\Gamma \to \mathbb{Q}$ is a rational valued function such that for each $g \in E_\Gamma$, we have $\beta(g) \geq 1$. A vertex $a \in V_\Gamma$ is called *smooth* or *artificial* if a is connected with exactly two edges and these edges connect a with exactly two vertices of Γ. A vertex a is called a *loop vertex* if a is connected with exactly two edges and these edges connect a with the same vertex of Γ.

An Abstract Hölder Complex (Γ, β) is called *Canonical* or *Simplified* if

(1) Γ has no artificial vertices;
(2) for any loop vertex connected with two edges g_1 and g_2 we have $\beta(g_1) = \beta(g_2)$.

The Standard Hölder Triangle T_β is a semialgebraic subset of $\mathbb{R}^2$ defined as follows:

$$T_\beta = \{(x,y) \in \mathbb{R}^2 \ : \ 0 \leq y \leq x^\beta, \ 0 \leq x \leq 1\},$$

where $\beta \geq 1$ is a rational number.

A semialgebraic set $X \subset \mathbb{R}^n$ is called a *Geometric Hölder Complex associated to* (Γ, β) if

(1) there exists a homeomorphism $\Phi\colon cone(\Gamma) \to X$, where $cone(\Gamma)$ is a cone over Γ;
(2) for any edge $g \in E_\Gamma$, the image of the set $cone(g) \subset cone(\Gamma)$ by the map Φ is semialgebraically bi-Lipschitz equivalent, with respect to the inner metric, to T_β, where $\beta = \beta(g)$. If $\Psi\colon \Phi(cone(g)) \to T_\beta$ is the corresponding bi-Lipschitz map, then $\Psi(x_0) = 0$, where $x_0 \in X$ is the image of the vertex of $cone(\Gamma)$ by the map Φ.

Theorem 2.1. [1] *Let $X \subset \mathbb{R}^n$ be a closed semialgebraic set of dimension 2 and let $x_0 \in X$. Then there exists a unique (up to isomorphism) Canonical Hölder Complex (Γ, β) such that, for sufficiently small $\epsilon > 0$, $X \cap B(x_0, \epsilon)$ is a Geometric Hölder Complex, associated to (Γ, β).*

Let $a_1 \geq a_2 \geq \cdots \geq a_n$ be a finite sequence of positive integer numbers. A *Weighted homogeneous foliation* $\mathcal{F}_{(a_1,\ldots,a_n)}$ in $\mathbb{R}^n$ with the weights $a_1 \geq a_2 \geq \cdots \geq a_n$ is a singular foliation defined as a family of curves $(t^{a_1}x_1, \ldots, t^{a_n}x_n)$, where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $t \in (0, +\infty)$. The *Standard Newton Simplex*, associated

to a weighted homogeneous foliation $\mathcal{F}_{(a_1,\dots,a_n)}$ is the convex hull of the points $(a_1, 0\dots,0),(0,a_2,0,\dots,0),\dots,(0,\dots,0,a_n)$. All the 1-dimensional faces of the Standard Newton Simplex belong to subspaces

$$\mathbb{R}^2_{ij} = span\{(0,\dots,a_i,0,\dots,0),(0,\dots,a_j,0,\dots,0)\}.$$

The quotient $\dfrac{a_i}{a_j}$ $(i<j)$ is called the *direction* of (i,j)-1-dimensional face of a Standard Newton Simplex.

Let $x \neq 0$ be a point in $\mathbb{R}^n$. Denote by γ_x the closure of the leaf of $\mathcal{F}_{(a_1,\dots,a_n)}$ passing through the point x. A set $X \subset \mathbb{R}^n$ is called $(a_1,\dots,a_n)$-*weighted homogeneous* if, for all $x \in X$, we have $\gamma_x \subset X$.

Example 2.2. Let $f\colon \mathbb{R}^3 \to \mathbb{R}$ be a weighted homogeneous polynomial with respect to weights $a \geq b \geq c$. Then $X = f^{-1}(0)$ is an (a,b,c)-weighted homogeneous algebraic set.

Example 2.3. Let $F\colon \mathbb{R}^m \to \mathbb{R}^n$ be a polynomial map with coordinate functions $F = (f_1,\dots,f_n)$ such that $f_1,\dots,f_n$ are weighted homogeneous polynomials with degrees $(d_1,\dots,d_n)$. Then $F(R^m)$ is a $(d_1,\dots,d_n)$-weighted homogeneous semialgebraic set.

Proposition 2.4. *Let $X \subset \mathbb{R}^n$ be a semialgebraic $(a_1,\dots,a_n)$-weighted homogeneous subset. Then $Sing(X)$ is also an $(a_1,\dots,a_n)$-weighted homogeneous set.*

Proof. Let us consider a map $\varphi_t\colon \mathbb{R}^n \to \mathbb{R}^n$ defined as follows:

$$\varphi_t(x_1,\dots,x_n) = (t^{a_1}x_1,\dots,t^{a_n}x_n).$$

Note that a restriction of this map to $\mathbb{R}^n - \{0\}$ is a diffeomorphism, for all $t>0$. If $X \subset \mathbb{R}^n$ is a semialgebraic $(a_1,\dots,a_n)$-weighted homogeneous subset, then, for all $t>0$, $\varphi_t(X) = X$. Thus, $\varphi_t(Sing(X)) = Sing(X)$ and hence $Sing(X)$ is $(a_1,\dots,a_n)$-weighted homogeneous set. □

A closed semialgebraic set $X \subset \mathbb{R}^n$ is called a *semialgebraic surface* if $\dim X = 2$.

Theorem 2.5. *Let $X \subset \mathbb{R}^n$ be a semialgebraic $(a_1,\dots,a_n)$-weighted homogeneous surface. Let (Γ,β) be the Canonical Hölder Complex of X at 0. Then, for each $g \in E_\Gamma$, $\beta(g) = 1$ or is one of the (i,j)-directions of the Standard Newton Simplex associated to $\mathcal{F}_{(a_1,\dots,a_n)}$.*

For the case of weighted homogeneous surfaces in $\mathbb{R}^3$, we will prove the following result.

Theorem 2.6. *Let $X \subset \mathbb{R}^3$ be a semialgebraic (a_1, a_2, a_3)-weighted homogeneous surface. If $0 \in X$ is an isolated singular point and the local link of X at 0 is connected, then the germ of X at 0 is bi-Lipschitz equivalent, with respect to the inner metric, to a germ at 0 of a β-horn, i.e. a surface defined as follows:*

$$H_\beta = \{(x_1, x_2, y) \in \mathbb{R}^3 \ : \ (x_1^2 + x_2^2) = y^{2\beta}\},$$

where β is equal to 1 or to $\dfrac{a_2}{a_3}$.

3. Order contact of semialgebraic arcs. Weighted homogeneous foliations

Recall that a semialgebraic arc γ at a point $x_0 \in \mathbb{R}^n$ is image of a semiagebraic map $\bar{\gamma}\colon [0, \epsilon) \to \mathbb{R}^n$ such that $\bar{\gamma}(0) = x_0$ and $\bar{\gamma}(s) \neq 0$, for $s \neq 0$. Let γ_1, γ_2 be two semialgebraic arcs at x_0. These arcs can be reparametrized near x_0 in the following form:

$$\gamma_i(t) = \{x \in \gamma_i \ : \ \|x - x_0\| = t\}; \ i = 1, 2.$$

Let $\rho(t) = \|\gamma_1(t) - \gamma_2(t)\|$. Since ρ is a semialgebraic function, we have

$$\rho(t) = at^\lambda + o(t^\lambda),$$

where λ is a rational number bigger or equal to 1 and $a > 0$. The number λ is called the *order of contact* of γ_1 and γ_2. We use the notation $\lambda(\gamma_1, \gamma_2)$.

Set

$$\widetilde{\gamma}_i(t) = \{x \in \gamma_i \ : \ \|x - x_0\|_{max} = t\}; \ i = 1, 2.$$

Let $\widetilde{\rho}(t) = \|\widetilde{\gamma}_1(t) - \widetilde{\gamma}_2(t)\|_{max}$. Recall that $\|x\|_{max} = max\{|x_1|, \dots, |x_n|\}$. Since $\widetilde{\rho}$ is also a semialgebraic function, we have

$$\widetilde{\rho}(t) = \widetilde{a}t^{\widetilde{\lambda}} + o(t^{\widetilde{\lambda}}),$$

where $\tilde{\lambda}$ is a rational number bigger or equal to 1 and $\tilde{a} > 0$.

Proposition 3.1. *The number $\widetilde{\lambda}$, defined above, is equal to $\lambda(\gamma_1, \gamma_2)$.*

We are going to prove this proposition in Section 6.

Proposition 3.2. [2]

Let γ_1, γ_2 and γ_3 be three semialgebraic arcs at $x_0 \in \mathbb{R}^n$. Let $\lambda(\gamma_1, \gamma_2) \geq \lambda(\gamma_2, \gamma_3) \geq \lambda(\gamma_1, \gamma_3)$, then $\lambda(\gamma_2, \gamma_3) = \lambda(\gamma_1, \gamma_3)$.

The main result of this section is the following.

Theorem 3.3. *Let $\mathcal{F}_{(a_1,\ldots,a_n)}$ be a weighted homogeneous foliation in $\mathbb{R}^n$. For all $x \neq y$ in $\mathbb{R}^n$, the order contact $\lambda(\gamma_x, \gamma_y)$ is equal to 1 or to a direction of a 1-dimensional face of the Standard Newton Simplex associated to $\mathcal{F}_{(a_1,\ldots,a_n)}$.*

Proof. Let us proceed by induction on n. First, we consider the case $n = 2$. In this case, all the leaves of this foliation can be presented in one of the following forms:

(1) $x_1 \geq 0$, $x_2 = a x_1^{\alpha}$, where $a \in \mathbb{R}$ and $\alpha = \dfrac{a_1}{a_2}$;
(2) $x_1 \leq 0$, $x_2 = a|x_1|^{\alpha}$, where $a \in \mathbb{R}$ and $\alpha = \dfrac{a_1}{a_2}$;
(3) $x_1 = 0$, $x_2 \leq 0$;
(4) $x_1 = 0$, $x_2 \geq 0$.

Using Proposition 3.1 one can show that $\lambda(\gamma_x, \gamma_y)$ is equal to 1 or to α.

Suppose that the statement is true for all the weighted homogeneous foliations $\mathcal{F}_{(a_1,\ldots,a_k)}$ in $\mathbb{R}^k$ for $k < n$. Consider a foliation $\mathcal{F}_{(a_1,\ldots,a_n)}$. Note that the restriction of $\mathcal{F}_{(a_1,\ldots,a_n)}$ to the hyperplane $x_n = 0$ is a weighted homogeneous foliation $\mathcal{F}_{(a_1,\ldots,a_{n-1})}$ in $\mathbb{R}^{n-1}$. Thus, for any two curves γ_y, γ_z belongs to the hyperplane, the statement is true, by the induction hypotheses. Thus, we can suppose that γ_y, γ_z are chosen in such a way that $z = (z_1, \ldots, z_n)$ and $z_n \neq 0$. Let $y = (y_1, \ldots, y_n)$. If $y_n = 0$, then the unit tangent vector at zero to γ_y belong to the hyperplane $x_n = 0$ and the unit tangent vector to γ_z does not belong to this hyperplane. Thus, $\lambda(\gamma_y, \gamma_z) = 1$. If y and z belong to different sides of the hyperplane $x_n = 0$, then their unit tangent vectors at zero cannot coincide. Again, in this case, $\lambda(\gamma_y, \gamma_z) = 1$. Now, we suppose that $z_n > 0$ and $y_n > 0$ (the case $z_n < 0$ and $y_n < 0$ can be treated in the same way). Consider the parametrization $\bar{\gamma}_y(t)$ of γ_y and $\bar{\gamma}_z(t)$ of γ_z defined in the beginning of this section. We have

$$\bar{\gamma}_y(t) = (t^{\frac{a_1}{a_n}} \bar{y}_1, \ldots, t^{\frac{a_{n-1}}{a_n}} \bar{y}_{n-1}, t)$$

and

$$\bar{\gamma}_z(t) = (t^{\frac{a_1}{a_n}} \bar{z}_1, \ldots, t^{\frac{a_{n-1}}{a_n}} \bar{z}_{n-1}, t)$$

where $\bar{y} = (\bar{y}_1, \ldots, \bar{y}_{n-1}, 1)$ and $\bar{z} = (\bar{z}_1, \ldots, \bar{z}_{n-1}, 1)$ are the intersections of γ_y and γ_z, respectively, with the hyperplane $x_n = 1$. We obtain

$$\widetilde{\rho}(t) = max\{t^{\frac{a_1}{a_n}}|\bar{y}_1 - \bar{z}_1|, \ldots, t^{\frac{a_{n-1}}{a_n}}|\bar{y}_{n-1} - \bar{z}_{n-1}|\}.$$

Hence, $\lambda(\gamma_y, \gamma_z)$ can be equal to $\dfrac{a_1}{a_n}, \ldots, \dfrac{a_{n-1}}{a_n}$. The theorem is proved. $\square$

Remark 3.4. By the construction, it is clear that, for all the pairs $(i > j)$, there exists a pair of curves γ_y and γ_z such that $\lambda(\gamma_y, \gamma_z) = \dfrac{a_i}{a_j}$.

4. Hölder exponents. Horn exponents.

A semialgebraic surface $X \subset \mathbb{R}^n$ is called a *β-Hölder Triangle* at $x_0 \in X$ if the germ of X at x_0 is semialgebraically bi-Lipschitz equivalent to a germ of the standard β-Hölder Triangle $T_\beta \subset \mathbb{R}^2$, with respect to the inner metric, and the image of the point x_0 by the corresponding bi-Lipschitz map is the point $(0,0) \in \mathbb{R}^2$. The inverse images of the boundary curves of T_β, containing $(0,0)$ are called *sides* of the β-Hölder Triangle X. The number β is called the *Hölder Exponent* of X at x_0. We use the notation $\beta(X, x_0)$.

A semialgebraic surface $X \subset \mathbb{R}^n$ is called a *β-Horn* at a point $x_0 \in X$ if the germ of X at x_0 is semialgebraically bi-Lipschitz equivalent to the germ at $(0,0,0) \in \mathbb{R}^3$ of the standard β-Horn, i.e. a semialgebraic set defined as follows:

$$H_\beta = \{(x_1, x_2, y) \in \mathbb{R}^3 \ : \ (x_1^2 + x_2^2) = y^{2\beta}\},$$

with respect to the inner metric, and the image of the point x_0 by the corresponding bi-Lipschitz map is the point $(0,0,0) \in \mathbb{R}^3$. The number β is called the *Horn Exponent* of X at x_0. We are going to use the same notation $\beta(X, x_0)$.

The following result is useful for calculations of Hölder Exponents and Horn Exponents.

Theorem 4.1. *Let $X \subset \mathbb{R}^n$ be a semialgebraic surface. Let $x_0 \in X$ be a point such that X is a β-Hölder Triangle at x_0 or a β-Horn at x_0. Then* $\beta(X, x_0) = \inf\{\lambda(\gamma_1, \gamma_2) \ : \ \gamma_1, \gamma_2$ *are semialgebraic arcs on X with* $\gamma_1(0) = \gamma_2(0) = x_0\}$.

Proof. We are going to prove the statement for a β-Hölder Triangle. The proof for a β-Horn is the same. Let $X \subset \mathbb{R}^n$ be a β-Hölder Triangle at $x_0 \in X$. By the

main result of [1] (see also [6]), there exists a finite set of semialgebraic arcs at x_0, $\{\gamma_1, \dots, \gamma_k\}$, $\gamma_i \subset X$ for all $i = 1, \dots k$, such that

(1) γ_i, γ_{i+1} are sides of a β_i-Hölder Triangle $X_i \subset X$, where $\beta_i = \lambda(\gamma_i, \gamma_{i+1})$;
(2) $\gamma_j \cap X_i = x_0$ if $j \neq i$ and $j \neq i + 1$;
(3) $X_i \cap B(x_0, \epsilon)$ is normally embedded in $\mathbb{R}^n$, for sufficiently small $\epsilon > 0$.

By the simplification theorem of [1], $\beta(X, x_0) = \min \beta_i$.

Let $\alpha_1, \alpha_2 \subset X$ be two semialgebraic arcs at x_0. Since α_1 and α_2 are semialgebraic, then there exist two subsets X_{j_1} and X_{j_2}, defined above, such that $\alpha_1 \subset X_{j_1}$ and $\alpha_2 \subset X_{j_2}$. We can suppose that $j_1 < j_2$. By Proposition 3.2, we obtain

$$\lambda(\alpha_1, \alpha_2) = \min\{\lambda(\alpha_1, \gamma_{j_1+1}), \lambda(\gamma_{j_1+1,\gamma_{j_1+2}}), \dots, \lambda(\gamma_{j_2}, \alpha_2)\}.$$

By the same reason,

$$\lambda(\alpha_1, \gamma_{j_1+1}) \geq \beta_{j_1} \text{ and } \lambda(\gamma_{j_2}, \alpha_2) \geq \beta_{j_2}.$$

By these three inequalities, we obtain $\lambda(\alpha_1, \alpha_2) \geq \beta(X, x_0)$.

On the other hand, there exists a pair γ_i, γ_{i+1}, such that $\beta(X, x_0) = \beta_i = \lambda(\gamma_i, \gamma_{i+1})$. □

5. Canonical Hölder Complex for weighted homogeneous surfaces.

This section is devoted to a proof of Theorem 2.5. We use induction on the dimension of the ambient space $\mathbb{R}^n$.

Let $X \subset \mathbb{R}^2$ be a closed semialgebraic surface which is (a_1, a_2)-weighted homogeneous. If $X \neq \mathbb{R}^2$, then X is a collection of some Hölder Triangles $X_1, \dots, X_p$ such that $X_i \cap X_j = \{0\}$ if $i \neq j$. By Proposition 2.4, we have two possibilities:

(1) the boundary curves of X_i belong to $Sing(X)$ and, thus, the boundary curves γ_1 and γ_2 are leaves of the weighted homogeneous foliation;
(2) X_i is a " half " of a weighted homogeneous β-Horn, in this case we can also suppose that the boundary curves of X_i are leaves of this foliation.

Thus, X_i is a (a_1, a_2)-weighted homogeneous set. If a β-Hölder Triangle X_i intersects with the set $x_2 = 0$ only at $\{0\}$, then $\beta(X_i, 0) = \frac{a_1}{a_2}$. Otherwise, $\beta(X_i, 0) = 1$. Let us observe that $\beta(\mathbb{R}^2, 0) = 1$. The first step of induction is done.

Let $X \subset \mathbb{R}^n$ be a semialgebraic $(a_1, \dots, a_n)$-weighted homogeneous surface. Let X_i be a β-Hölder Triangle corresponding to the Canonical Complex of X at 0. If

X_i belongs to the hyperplane $x_n = 0$, then the statement is true, by the induction hypothesis. Thus, let us suppose that X_i does not belong to the hyperplane $x_n = 0$, i.e. there exists a curve $\gamma_z \subset X_i$ such that $\gamma_z \cap \{x_n = 0\} = \{0\}$. Now, if $X_i \cap \{x_n = 0\} \neq \{0\}$, then there exists a curve $\gamma_y \subset X_i \cap \{x_n = 0\}$. The curves γ_z and γ_y have different unit tangent vectors at $0 \in \mathbb{R}^n$ and, thus, $\beta(X_i, 0) = 1$. Now we consider the case when $X_i \cap \{x_n = 0\} = \{0\}$. We are going to show that, for any pair of semialgebraic arcs $\alpha_1, \alpha_2 \subset X_i$ with same initial point $0 \in \mathbb{R}^n$, there exists a pair of leaves γ_{z_1} and γ_{z_2} such that $\lambda(\alpha_1, \alpha_2) \geq \lambda(\gamma_{z_1}, \gamma_{z_2})$. In order to prove this statement, we need the following lemma.

Lemma 5.1. *Let $Y \subset \mathbb{R}^n$ be a $(a_1, \ldots, a_n)$ weighted homogeneous surface, such that $Y - \{0\}$ is connected. Let $Y \cap \{x_n = 0\} = \{0\}$. Suppose that the section $Y \cap \{x_n = 1\}$ is contained in the plane $\{x_i = r\}$. Then, for every positive value ϵ, there exists a value $r(\epsilon)$ such that the section $Y \cap \{x_n = \epsilon\}$ is contained to the plane $x_i = r(\epsilon)$.*

Proof. Take $r(\epsilon) = r\epsilon^{\frac{a_i}{a_n}}$. □

Let $M = X_i \cap \{x_n = 1\}$. Suppose that there exists an index k such that $M \subset \{x_k = r\}$. Let us consider a projection $P \colon \mathbb{R}^n \to \mathbb{R}^{n-1}$ defined as follows:

$$P(x_1, \ldots, x_n) = (x_1, \ldots, \hat{x_k}, \ldots, x_n).$$

Observe that, for $j \neq k$, $P|_{X_j}$ is a bi-Lipschitz map and $P(X_j)$ is a $(a_1, \ldots, \hat{a_k}, \ldots, a_n)$-weighted homogeneous subset in $\mathbb{R}^{n-1}$. Then we obtain our statement from the induction hypotheses.

Now, let us suppose that $M \not\subset \{x_{n-1} = r\}$, for all r. Let $T \colon M \times [0, +\infty) \to X_i$ be a map defined as follows:

$$T(x_1, \ldots, x_n, t) = (t^{\frac{a_1}{a_n}} x_1, \ldots, t^{\frac{a_{n-1}}{a_n}} x_{n-1}, t).$$

Clearly, the map T is semialgebraic, injective and surjective, for $t \neq 0$.

Let $\alpha \colon [0, \rho] \to X_i$ be a semialgebraic arc in X_i such that $\alpha(0) = 0$, parameterized in such a way that $\alpha'(0)$ exists. Let $\bar{\alpha} \colon [0, \rho] \to M \times [0, 1]$ be a lifting of α, i.e. $T \circ \bar{\alpha} = \alpha$. This lifting is defined as follows: $\bar{\alpha}(s) = T^{-1}(\alpha(s))$, for $s \neq 0$. Since $\bar{\alpha}(s)$ is semialgebraic, then $\lim\limits_{s \to 0} \bar{\alpha}(s)$ exists and belongs to $M \times \{0\}$. By the same reason, $\lim\limits_{s \to 0} \bar{\alpha}'(s)$ also exists. Therefore, the arc $\bar{\alpha}$ near $M \times \{0\}$ can be reparameterized

in the following way: $\bar{\alpha}(t) = (x_1(t), \ldots, x_{n-1}(t), t)$. By definition of the map T, we have

$$\alpha(t) = (t^{\frac{a_1}{a_n}} x_1(t), \ldots, t^{\frac{a_{n-1}}{a_n}} x_{n-1}(t), t).$$

Clearly, $\frac{d}{dt}|_{t=0}(\alpha(t))$ is not contained in the hyperplane $x_n = 0$.

Let $\alpha_1, \alpha_2 \colon [0, \rho] \to X_i$ be two arcs such that $\alpha_1(0) = \alpha_2(0) = 0$. The arcs α_1 and α_2 can be parameterized as follows:

$$\alpha_1(t) = (t^{\frac{a_1}{a_n}} y_1(t), \ldots, t^{\frac{a_1}{a_{n-1}}} y_{n-1}(t), t) \text{ and } \alpha_2(t) = (t^{\frac{a_1}{a_n}} z_1(t), \ldots, t^{\frac{a_1}{a_{n-1}}} z_{n-1}(t), t).$$

We obtain:

$$\|\alpha_1(t) - \alpha_2(t)\|_{max} = max\{t^{\frac{a_i}{a_n}} |y_i(t) - z_i(t)|;\ i = 1, \ldots, n-1\}.$$

Since $|y_i(t) - z_i(t)|$ is a bounded function and $a_1 \geq \cdots \geq a_{n-1}$, we have

$$\|\alpha_1(t) - \alpha_2(t)\|_{max} = at^{\lambda} + o(t^{\lambda})$$

with $\lambda \geq \frac{a_{n-1}}{a_n}$.

On the other hand, since $M \not\subset \{x_{n-1} = r\}$, there exist $y = (y_1, \ldots, y_{n-1}, 1), z = (z_1, \ldots, z_{n-1}, 1) \in M$ such that $y_{n-1} \neq z_{n-1}$. By Theorem 3.3, the leaves γ_y and γ_z have the order of contact $\lambda(\gamma_y, \gamma_z) = \frac{a_{n-1}}{a_n}$.

The theorem is proved. $\square$

Proof of Theorem 2.6. Let $X \subset \mathbb{R}^3$ be a semialgebraic (a_1, a_2, a_3)-weighted homogeneous surface with a connected local link at 0. If $X \cap \{x_3 = 0\} \neq \{0\}$, then, by the proof of the Theorem 2.5, we obtain that $\beta(X, 0) = 1$.

Note that $\beta(X, 0)$ can be equal to $\frac{a_1}{a_3}$ only in the case that $X \cap \{x_3 = \epsilon\}$ is totally included in a line $x_2 = r(\epsilon)$. But, since the local link of X at 0 is connected, it implies that $X \cap \{x_3 = \epsilon\}$ is the set defined by $x_3 = \epsilon$, $x_2 = r(\epsilon)$. Then X is a union of standard leaves of $\mathcal{F}_{(a_1,a_2,a_3)}$ passing through the points belonging to the straight line $x_3 = \epsilon$, $x_2 = r(\epsilon)$. Note, that if $X \cap \{x_3 = 0\} = \{0\}$ then X cannot be closed.

The case $\beta(X, 0) = \frac{a_1}{a_2}$ can occur if, and only if, $X \subset \{x_3 = 0\}$. But, in this case, $\beta(X, 0) = 1$. $\square$

6. Order comparison lemma

Let K be a field of germs of subanalytic functions $f\colon (0,\epsilon)\to\mathbb{R}$. Let $\nu\colon K\to\mathbb{R}$ be a canonical valuation on K. Namely, if $f(t)=\alpha t^\beta+o(t^\beta)$ with $\alpha\neq 0$, we put $\nu(f)=\beta$.

Here we are going to prove a bit more general result such that Lemma 3.1 is a partial case of it.

Theorem 6.1. *Let $\|\cdot\|_S$ be a semialgebraic norm on $\mathbb{R}^n$. Let γ_1 and γ_2 be two semi-analytic arcs such that $\gamma_1(0)=\gamma_2(0)=x_0\in\mathbb{R}^n$. Let $\gamma_i^S(t)$ be a parametrization of γ_i such that $\|\gamma_i^S(t)-x_0\|_S=t$, $i=1,2$. Let $\lambda_S(\gamma_1,\gamma_2)=\nu(\|\gamma_1^S(t)-\gamma_2^S(t)\|_S)$. Then $\lambda_S(\gamma_1,\gamma_2)=\lambda(\gamma_1,\gamma_2)$.*

In order to prove this theorem we need the following lemma.

Lemma 6.2. *Let $M\subset\mathbb{R}^n$ be a semialgebraic convex compact subset such that $0\in Int(M)$. Then, for each small $\epsilon>0$, there exists a number $\delta>0$ such that, for each pair $x,y\in\partial M$ with $\|x-y\|<\epsilon$, the angle between x and $x-y$ satisfies the following inequality*

$$\delta<\angle(x,x-y)<\pi-\delta.$$

Proof. Let $Tang(M)$ be a subset of $\mathbb{R}^n\times\mathbb{R}P^{n-1}$ of the pairs (x,l) such that l is a straight line, $l\cap Int(M)=\emptyset$ and $x\in\partial M\cap l$. Clearly, $Tang(M)$ is a compact semialgebraic subset of $\mathbb{R}^n\times\mathbb{R}P^{n-1}$. Let $ang\colon\mathbb{R}^n\times\mathbb{R}P^{n-1}\to\mathbb{R}$ be a function defined as follows:

$$ang(x,l)=\sin(\angle(\overrightarrow{0x},l)).$$

Observe that $\sin(\angle(\overrightarrow{0x},l))$ is a well defined function. Since $Tang(M)$ is compact, then there exists $\tilde{\delta}>0$ such that, for all $(x,l)\in Tang(M)$, we have $ang(x,l)>\tilde{\delta}$.

Let $Tang_\epsilon(M)$ be a set of pairs (x,l) where $x\in\partial M$ and l is a straight line passing through x and some $y\in\partial M$ such that $\|x-y\|\leq\epsilon$. Observe that $Tang_\epsilon(M)$ is also a compact semialgebraic set. Since the Hausdorff limit $\lim\limits_{\epsilon\to 0}Tang_\epsilon(M)$ belongs to $Tang(M)$, there exists $\tilde{\epsilon}>0$ such that $ang(x,l)>\dfrac{\tilde{\delta}}{2}$, for all $(x,l)\in Tang_{\tilde{\epsilon}}(M)$. It proves the lemma. □

Proof of Theorem 6.1. Let x_0 and let γ_1,γ_2 be arcs satisfying the condition of the theorem. Let us prove that $\lambda_S(\gamma_1,\gamma_2)\geq\lambda(\gamma_1,\gamma_2)$. Suppose that $\lambda_S(\gamma_1,\gamma_2)<$

$\lambda(\gamma_1, \gamma_2)$. Let $\gamma_1(t)$ and $\gamma_2(t)$ be points such that $\|\gamma_1(t)\| = \|\gamma_2(t)\| = t$. Let $\tau = |\gamma_1(t)\|_S$. Let $\gamma_2^S(\tau)$ be a point on γ_2 such that $\|\gamma_2^S(\tau)\|_S = \tau$. Thus, for small t, the angle at the vertex $\gamma_2^S(\tau)$ of the triangle $\gamma_1(t), \gamma_2(t), \gamma_2^S(\tau)$ must tend to zero. The line defined by $\gamma_2(t)$ and $\gamma_2^S(\tau)$ tends to the tangent line of γ_2 at 0. Since the ball of radius τ, with respect to the norm $\| \cdot \|_S$, is a convex set and the origin belongs to this ball, we obtain a contradiction to Lemma 6.2.

Using the similar argument we can show that $\lambda_S(\gamma_1, \gamma_2) \leq \lambda(\gamma_1, \gamma_2)$. □

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